Time evolution of a two-atom dressed entangled state in a cavity

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We study the time evolution of superposition of product states of two dressed atoms in a spherical cavity in the extreme situations of an arbitrarily large cavity (free space) and of a small one. In the large-cavity case, the system dissipates, whereas, for the small finite cavity, the system evolves in an oscillating way and never completely decays. We also compute the von Neumann entropy for such a system, a measurement of the degree of entanglement of the two atoms, as the superposed state evolves in time. We find that this entropy does not depend on time, nor on the size of the cavity.

I. INTRODUCTION

In the quantum mechanical description of multipartite systems, with Hilbert spaces given by direct products of individual part spaces, the superposition principle leads naturally to entangled states which can not be written as single products of states of the constituent parts; non interacting subsystems can thus share entangled states that hold quantum correlations. Such quantum entanglement carries nonlocal features which can be analyzed by comparison with classical correlations [1, 2].

Entanglement is a quantum mechanical resource that plays a crucial role in implementing teleportation of quantum states and in several applications of quantum computation and quantum information [3, 4, 5]. Quantifying entanglement then becomes an important issue which has been addressed in the literature from a variety of viewpoints [6, 7, 8, 9, 10, 11, 12]. For bipartite systems, the measurement of entanglement is well established, the von Neumann entropy of the reduced density matrix providing the simplest measure of the degree of entanglement of a given state. In this way, maximum entangled states has been constructed for both boson [13, 14] and fermion [15] bipartite systems.

In a recently reported experiment [16], it is proven the existence of deterministic entanglement of separated oscillators, consisting of the vibrational states of two pairs of atomic ions in different locations. They also demonstrate entanglement of the internal states of an atomic ion with a distant mechanical oscillator. The authors claim that such experiments may lead to the generation of entangled states of mechanical oscillators in a larger scale, in such a way as to provide tests for nonlocality in mesoscopic systems. They also claim that these experiments could be used to control quantum information processing based on trapped atomic ions. Previously, an experiment was performed in [17], using ultraviolet lasers to entangle two pairs of beryllium ions in an electromagnetic trap. These authors also cross-entangled the entangled pairs, that is, entangled each member of the first pair with its correspondent in the second pair. Then the first pair of ions was measured, and the results were used as an indication of whether the unmeasured second pair was entangled.

In the present paper we study the time evolution of an entangled two-atom state, in the presence of a force field. Our approach to this problem makes use of the concept of dressed states. This formalism, originally introduced in [18], was already employed to investigate several situations [19, 20, 21, 22, 23, 24, 25]. It accounts for the fact that, for instance, a charged physical particle is always coupled to the force (gauge) field; in other words, it is always "dressed" by a cloud of quanta of the gauge field. In general for a system of matter particles, the idea is that the particles are coupled to an environment, which is usually modeled in two equivalent ways: either to represent it by a free field, as was done in Refs. [26, 27], or to consider the environment as a reservoir composed of a large number of noninteracting harmonic oscillators (see, for instance, [28, 29, 30, 31]). In both cases, exactly the same type of argument given above in the case of a charged particle applies, with the appropriate changes, to such systems. We may then speak of the "dressing" of the set of particles by the ensemble of the harmonic modes of the environment. It should be true in general for any system in which material particles are coupled to an environment. In atomic physics, the semiqualitative idea of a "dressed atom" has been largely employed in studies involving the interaction of atoms and electromagnetic fields [32]. In the realm of general physics, the dressing of a matter particle by an environment has found an application in describing the radiation damping of classical systems [33]. Our dressed states can be viewed as a rigorous version of these dressing procedures, in the context of the model employed here.

We will consider our system in this paper as consisting of two atoms, each one of them interacting independently inside a spherical cavity with an environment provided by the harmonic modes of a field. We take it as a bipartite system, each subsystem consisting of one of the dressed atoms. We will consider a superposition of two kinds of states:

either all entities (both atoms and the field modes) are in their ground states, or just one of the atoms lies in its first excited state, the other one and all the field modes being in their ground states. The analysis of the (reduced) density matrix of the system leads to the computation of the von Neumann entropy, which measures the degree of entanglement of the two atoms.

The dressing formalism for just one atom inside a cavity is briefly reviewed in Section 2 in order to establish basic notation and formulas for the time evolution of the states. In Section 3 the formalism is generalized for the two-atom system and describe the evolution of its density matrix, either in the case of a very large cavity (with infinite radius, that is, free space) or of a small cavity. The entanglement of the two atoms is discussed in Section 4. Finally, in Section 5 we present our conclusions.

II. A SINGLE DRESSED ATOM

Before tackling the case of two atoms, it is convenient to reproduce here the analysis of Ref. [24] for the simpler situation of just one atom, dressed by its interaction with the environment field. We present, in this section, a short review of the formalism introduced in previous works.

We shall thus consider a atom in the harmonic approximation, coupled linearly to an environment modeled by the infinite set of harmonic modes of a scalar field, on the inside a spherical cavity. A nonperturbative study of the time evolution of such a system is implemented by means of *dressed* states and *dressed* coordinates [18]. In particular, our dressed states are *not* the same as those currently employed in the literature, usually associated to normal coordinates. Our dressed states are given in terms of our dressed coordinates and allow a rigorous study of the time evolution of quantum systems in the context of the model employed here. The results we obtain by these means are those expected on physical grounds, but contain corrections with respect to the formulas obtained from perturbation theory.

Let us start by considering an atom labeled λ, having *bare* frequency ω_λ, linearly coupled to a field described by N ($\to \infty$) other oscillators, with frequencies ω_k, $k = 1, 2, \ldots, N$. The whole system is contained in a perfectly reflecting spherical cavity of radius R, the free space corresponding to the limit $R \to \infty$. Hereafter, we shall refer to the harmonic oscillator as the *atom*, to distinguish it from the harmonic modes of the environment. Denoting by $q_\lambda(t)$ ($p_\lambda(t)$) and $q_k(t)$ ($p_k(t)$) the coordinates (momenta) associated with the atom and the field oscillators, respectively, the Hamiltonian of the system is taken as

$$H_\lambda = \frac{1}{2}\left[p_\lambda^2 + \omega_\lambda^2 q_\lambda^2 + \sum_{k=1}^{N}\left(p_k^2 + \omega_k^2 q_k^2\right)\right] - q_\lambda \sum_{k=1}^{N} \eta_\lambda \omega_k q_k, \tag{1}$$

where η_λ is a constant and the limit $N \to \infty$ will be understood later on. The Hamiltonian (1) can be turned to principal axis by means of a point transformation,

$$q_{\mu(\lambda)} = \sum_{r_\lambda=0}^{N} t_{\mu(\lambda)}^{r_\lambda} Q_{r_\lambda} \ , \quad p_{\mu(\lambda)} = \sum_{r_\lambda=0}^{N} t_{\mu(\lambda)}^{r_\lambda} P_{r_\lambda} \ , \tag{2}$$

where $\mu(\lambda) = (\lambda, \{k\})$, $k = 1, 2, \ldots, N$, and $r_\lambda = 0, \ldots, N$, performed by an orthonormal matrix $T = (t_{\mu(\lambda)}^{r_\lambda})$. The subscripts $\mu = \lambda$ and $\mu = k$ refer respectively to the atom and the harmonic modes of the field and r_λ refers to the normal modes. In terms of normal momenta and coordinates, the transformed Hamiltonian reads

$$H_\lambda = \frac{1}{2}\sum_{r_\lambda=0}^{N}\left(P_{r_\lambda}^2 + \Omega_{r_\lambda}^2 Q_{r_\lambda}^2\right), \tag{3}$$

where the Ω_{r_λ}'s are the normal frequencies corresponding to the collective *stable* oscillation modes of the coupled system.

Using the coordinate transformation $q_{\mu(\lambda)} = \sum_{r_\lambda} t_{\mu(\lambda)}^{r_\lambda} Q_{r_\lambda}$ in the equations of motion and explicitly making use of the normalization condition

$$\sum_{\mu=0}^{N}\left(t_{\mu(\lambda)}^{r_\lambda}\right)^2 = 1, \tag{4}$$

we get

$$t_k^{r_\lambda} = \frac{\eta_\lambda \omega_k}{\omega_k^2 - \Omega_{r_\lambda}^2} t_\lambda^{r_\lambda} \ , \quad t_\lambda^{r_\lambda} = \left[1 + \sum_{k=1}^{N} \frac{\eta_\lambda^2 \omega_k^2}{(\omega_k^2 - \Omega_{r_\lambda}^2)^2}\right]^{-\frac{1}{2}} , \tag{5}$$

with the condition

$$\omega_\lambda^2 - \Omega_{r_\lambda}^2 = \sum_{k=1}^{N} \frac{\eta_\lambda^2 \omega_k^2}{\omega_k^2 - \Omega_{r_\lambda}^2}. \tag{6}$$

The right-hand side of equation (6) diverges in the limit $N \to \infty$. Defining the counterterm $\delta\omega^2 = N\eta_\lambda^2$, it can be rewritten in the form

$$\omega_\lambda^2 - \delta\omega^2 - \Omega_{r_\lambda}^2 = \eta_\lambda^2 \Omega_{r_\lambda}^2 \sum_{k=1}^{N} \frac{1}{\omega_k^2 - \Omega_{r_\lambda}^2}. \tag{7}$$

Equation (7) has $N+1$ solutions, corresponding to the $N+1$ normal collective modes. It can be shown [18] that if $\omega_\lambda^2 > \delta\omega^2$, all possible solutions for Ω^2 are positive, physically meaning that the system oscillates harmonically in all its modes. On the other hand, when $\omega_\lambda^2 < \delta\omega^2$, one of the solutions is negative and so no stationary configuration is allowed.

Therefore, we just consider the situation in which all normal modes are harmonic, which corresponds to the first case above, $\omega_\lambda^2 > \delta\omega^2$, and define the *renormalized* frequency

$$\bar{\omega}_\lambda^2 = \lim_{N\to\infty} \left(\omega_\lambda^2 - N\eta_\lambda^2\right), \tag{8}$$

following the pioneering work of Ref. [35]. In the limit $N \to \infty$, equation (7) becomes

$$\bar{\omega}_\lambda^2 - \Omega^2 = \eta_\lambda^2 \sum_{k=1}^{\infty} \frac{\Omega^2}{\omega_k^2 - \Omega^2}. \tag{9}$$

We see that, in this limit, the above procedure is exactly the analogous of mass renormalization in quantum field theory: the addition of a counterterm $-N\eta_\lambda^2 q_\lambda^2$ ($N \to \infty$) allows one to compensate the infinity of ω_λ^2 in such a way as to leave a finite, physically meaningful, renormalized frequency $\bar{\omega}_\lambda$.

To proceed, we take the constant η_λ as

$$\eta_\lambda = \sqrt{\frac{4g_\lambda \Delta\omega}{\pi}}, \tag{10}$$

where $\Delta\omega$ is the interval between two neighboring field frequencies and g is the coupling constant with dimension of frequency. The environment frequencies ω_k can be written in the form

$$\omega_k = k\frac{\pi c}{R}, \quad k = 1, 2, \ldots, \tag{11}$$

and, so, $\Delta\omega = \pi c/R$. Then, using the identity

$$\sum_{k=1}^{\infty} \frac{1}{k^2 - u^2} = \frac{1}{2}\left[\frac{1}{u^2} - \frac{\pi}{u}\cot(\pi u)\right], \tag{12}$$

equation (9) can be written in closed form:

$$\cot\left(\frac{R\Omega}{c}\right) = \frac{\Omega}{2g_\lambda} + \frac{c}{R\Omega}\left(1 - \frac{R\bar{\omega}_\lambda^2}{2g_\lambda c}\right). \tag{13}$$

The elements of the transformation matrix, turning the atom–field system to principal axis, are obtained in terms of the physically meaningful quantities Ω_{r_λ} and $\bar{\omega}_\lambda$ after some rather long but straightforward manipulations [18]. They read

$$t_\lambda^{r_\lambda} = \frac{\eta_\lambda \Omega_{r_\lambda}}{\sqrt{\left(\Omega_{r_\lambda}^2 - \bar{\omega}_\lambda^2\right)^2 + \frac{\eta_\lambda^2}{2}\left(3\Omega_{r_\lambda}^2 - \bar{\omega}_\lambda^2\right) + 4g_\lambda^2\Omega_{r_\lambda}^2}}, \tag{14}$$

$$t_k^{r_\lambda} = \frac{\eta_\lambda \omega_k}{\omega_k^2 - \Omega_{r_\lambda}^2} t_\lambda^{r_\lambda}. \tag{15}$$

Let us now consider the eigenstates of the system atom(λ)-field, $|l_\lambda, l_1, l_2, ...\rangle$, represented by the normalized eigenfunctions, written in terms of the normal coordinates $\{Q_{r_\lambda}\}$,

$$\phi_{l_\lambda l_1 l_2 ...}(Q,t) = \prod_s \left[\sqrt{\frac{2^{l_{s_\lambda}}}{l_{s_\lambda}!}} H_{l_{s_\lambda}} \left(\sqrt{\frac{\Omega_{s_\lambda}}{\hbar}} Q_{s_\lambda} \right) \right] \Gamma_0^\lambda \, e^{-i \sum_{s_\lambda} \left(l_{s_\lambda} + \frac{1}{2} \right) \Omega_{s_\lambda} t}, \tag{16}$$

where $H_{l_{s_\lambda}}$ stands for the l_{s_λ}-th Hermite polynomial and

$$\Gamma_0^\lambda = \mathcal{N}_\lambda e^{-\sum_s \frac{\Omega_{s_\lambda} Q_{s_\lambda}^2}{2}} \tag{17}$$

is the normalized vacuum eigenfunction, $\mathcal{N}_\lambda$ being the normalization factor.

We introduce *dressed* coordinates q'_λ and $\{q'_i\}$ for the *dressed* atom and the *dressed* field, respectively, defined by

$$\sqrt{\bar{\omega}_{\mu(\lambda)}} q'_{\mu(\lambda)} = \sum_{r_\lambda} t^{r_\lambda}_{\mu(\lambda)} \sqrt{\Omega_{r_\lambda}} Q_{r_\lambda}, \tag{18}$$

where $\bar{\omega}_{\mu(\lambda)} = \{\bar{\omega}_\lambda, \ \omega_i\}$. In terms of the dressed coordinates, we define for a fixed instant, $t = 0$, *dressed* states, $|\kappa_\lambda, \kappa_1, \kappa_2, \cdots\rangle$ by means of the complete orthonormal set of functions [18]

$$\psi_{\kappa_\lambda \kappa_1 ...}(q') = \prod_{\mu(\lambda)} \left[\sqrt{\frac{2^{\kappa_{\mu(\lambda)}}}{\kappa_{\mu(\lambda)}!}} H_{\kappa_{\mu(\lambda)}} \left(\sqrt{\frac{\bar{\omega}_{\mu(\lambda)}}{\hbar}} q'_{\mu(\lambda)} \right) \right] \Gamma_0^\lambda, \tag{19}$$

where $\mu(\lambda)$ labels collectively the dressed atom λ and the field modes, $1, 2, 3, \ldots$, $q'_{\mu(\lambda)} = q'_\lambda, \{q'_i\}$. The ground state Γ_0^λ in the above equation is the same as in equation (16). The invariance of the ground state is due to our definition of dressed coordinates given by equation (18). Notice that the introduction of the dressed coordinates implies, differently from the bare vacuum, the stability of the dressed vacuum state since, by construction, it is identical to the ground state of the interacting Hamiltonian (3). Each function $\psi_{\kappa_\lambda \kappa_1 ...}(q')$ describes a state in which the dressed oscillator q'_μ is in its $\kappa_{\mu(\lambda)}$-th excited state.

Let us consider the particular dressed state $\left|\Gamma_1^{\mu(\lambda)}(0)\right\rangle$ at $t = 0$, represented by the wave function $\psi_{00\cdots 1(\mu)0\cdots}(q')$. It describes the configuration in which *only* the μ-th dressed oscillator is in the *first* excited level, all other being in their ground states. As shown in Ref. [18], the time evolution of the state $\left|\Gamma_1^{\mu(\lambda)}\right\rangle$ is given by

$$\left|\Gamma_1^{\mu(\lambda)}(t)\right\rangle = \sum_\nu f_{\mu\nu}(t) \left|\Gamma_1^{\nu(\lambda)}(0)\right\rangle, \tag{20}$$

where $\mu(\lambda), \nu(\lambda) = \lambda, \{i\}$, with $\{i\}$ referring to the field modes, and

$$f_{\mu\nu}(t) = \sum_s t^s_\mu t^s_\nu e^{-i\Omega_s t}. \tag{21}$$

Moreover, it can be shown that, for all μ,

$$\sum_\nu |f_{\mu\nu}(t)|^2 = 1, \tag{22}$$

which allows to interpret the coefficients $f_{\mu\nu}(t)$ as probability amplitudes; for example, $f_{\lambda\lambda}(t)$ is the probability amplitude that, if the dressed atom is in the first excited state at $t = 0$, it remains excited at time t, while $f_{\lambda i}(t)$ represents the probability amplitude that the i-th dressed harmonic mode of the field be at the first excited level.

III. TIME EVOLUTION OF A DRESSED TWO-ATOM STATE

We now consider a bipartite system composed of two subsystems, $\mathcal{A}$ and $\mathcal{B}$; the subsystems consist respectively of dressed atoms A and B, in the sense defined in the preceding section, the whole system being contained in a perfectly reflecting sphere of radius R. Let us consider the eigenstates of the subsystems $\mathcal{A}$ and $\mathcal{B}$ with $\lambda = A, B$ labeling the quantities referring to the subsystems.

We consider the Hilbert space spanned by the dressed Fock-like states,

$$\left|\Gamma^{(AB)}_{n_A k_1 k_2 \cdots;\, n_B q_1 q_2 \cdots}\right\rangle \equiv |n_A, k_1, k_2, \ldots;\, n_B, q_1, q_2, \ldots\rangle = \left|\Gamma^{A}_{n_A, k_1, k_2, \ldots}\right\rangle \otimes \left|\Gamma^{B}_{n_B, q_1, q_2, \ldots}\right\rangle, \tag{23}$$

in which the dressed atom A is at the n_A excited level and the atom B is at the n_B excited level; the (doubled) dressed modes of the field are at the $k_1, k_2, \ldots$, $q_1, q_2, \ldots$ excited levels. Using this definition, let us consider at time $t = 0$, a family of entangled states of the bipartite system given by

$$\begin{aligned} |\Psi\rangle &= \sqrt{\xi}\, \left|\Gamma^{(AB)}_{1(A)00\cdots;0(B)00\cdots}(0)\right\rangle + \sqrt{1-\xi}\, e^{i\phi}\, \left|\Gamma^{(AB)}_{0(A)00\cdots;\,1(B)00\cdots}(0)\right\rangle \\ &= \sqrt{\xi}\, |1_A, 0, 0, \cdots; 0_B, 0, 0, \cdots\rangle + \sqrt{1-\xi}\, e^{i\phi}\, |0_A, 0, 0, \cdots;\, 1_B, 0, 0, \cdots\rangle, \end{aligned} \tag{24}$$

where $0 < \xi < 1$. In equation (24), $\left|\Gamma^{(AB)}_{1(A)0(B)00\cdots}(0)\right\rangle$ and $\left|\Gamma^{(AB)}_{0(A)1(B)00\cdots}(0)\right\rangle$ stand respectively for the states in which the dressed atom A (B) is at the first level, the dressed atom B (A) and all the field modes being in the ground state. They are

$$\left|\Gamma^{(AB)}_{1(A)0(B)00\cdots}(0)\right\rangle = \left|\Gamma^{A}_{100\cdots}(0)\right\rangle \otimes \left|\Gamma^{B}_{000\cdots}(0)\right\rangle \tag{25}$$

and

$$\left|\Gamma^{(AB)}_{0(A)1(B)00\cdots}(0)\right\rangle = \left|\Gamma^{A}_{000\cdots}\right\rangle \otimes \left|\Gamma^{B}_{100\cdots}(0)\right\rangle. \tag{26}$$

The density matrix at $t = 0$ is

$$\begin{aligned} \varrho(0) &= |\Psi\rangle\langle\Psi| \\ &= \xi\, |1_A, 0, 0, \cdots; 0_B, 0, 0, \cdots\rangle\langle 1_A, 0, 0, \cdots; 0_B, 0, 0, \cdots| \\ &\quad + (1-\xi)\, |0_A, 0, 0, \cdots;\, 1_B, 0, 0, \cdots\rangle\langle 0_A, 0, 0, \cdots;\, 1_B, 0, 0, \cdots| \\ &\quad + \sqrt{\xi(1-\xi)} e^{-i\phi}\, |1_A, 0, 0, \cdots; 0_B, 0, 0, \cdots\rangle\langle 0_A, 0, 0, \cdots;\, 1_B, 0, 0, \cdots| \\ &\quad + \sqrt{\xi(1-\xi)} e^{i\phi}\, |0_A, 0, 0, \cdots;\, 1_B, 0, 0, \cdots\rangle\langle 1_A, 0, 0, \cdots; 0_B, 0, 0, \cdots|, \end{aligned} \tag{27}$$

At time t, the state of the system is described by the density matrix

$$\varrho(t) = e^{-iHt}\, |\Psi\rangle\langle\Psi|\, e^{iHt}, \tag{28}$$

where H is the Hamiltonian of the whole system, such that

$$e^{-iHt} = e^{-iH_A t} \otimes e^{-iH_B t}$$

and H_A and H_B are the Hamiltonian H_λ of equations (1) or (3). We then obtain

$$\begin{aligned} \varrho(t) &= \xi \left(\left|\Gamma^{A}_{100\cdots}(t)\right\rangle\left\langle\Gamma^{A}_{100\cdots}(t)\right|\right) \otimes \left(\left|\Gamma^{B}_{000\cdots}\right\rangle\left\langle\Gamma^{B}_{000\cdots}\right|\right) \\ &\quad + (1-\xi) \left(\left|\Gamma^{A}_{000\cdots}\right\rangle\left\langle\Gamma^{A}_{000\cdots}\right|\right) \otimes \left(\left|\Gamma^{B}_{100\cdots}(t)\right\rangle\left\langle\Gamma^{B}_{100\cdots}(t)\right|\right) \\ &\quad + \sqrt{\xi(1-\xi)} e^{i\phi} \left(\left|\Gamma^{A}_{000\cdots}\right\rangle\left\langle\Gamma^{A}_{100\cdots}(t)\right|\right) \otimes \left(\left|\Gamma^{B}_{100\cdots}(t)\right\rangle\left\langle\Gamma^{B}_{000\cdots}\right|\right) \\ &\quad + \sqrt{\xi(1-\xi)} e^{-i\phi} \left(\left|\Gamma^{A}_{100\cdots}(t)\right\rangle\left\langle\Gamma^{A}_{000\cdots}\right|\right) \otimes \left(\left|\Gamma^{B}_{000\cdots}\right\rangle\left\langle\Gamma^{B}_{100\cdots}(t)\right|\right), \end{aligned} \tag{29}$$

where the states $\left|\Gamma^{A}_{000\cdots}\right\rangle$, $\left|\Gamma^{B}_{000\cdots}\right\rangle$ are stationary and the states $\left|\Gamma^{A}_{100\cdots}(t)\right\rangle$, $\left|\Gamma^{B}_{100\cdots}(t)\right\rangle$ evolve according to equation (20).

In order to investigate how the superposed states evolve in time, we shall consider the reduced density matrix obtained by tracing over all the degrees of freedom associated with the field. The computation is analogous to the one presented in Ref. [24]. After taking the trace, the density matrix has the indices referring to the 2-atom states. Explicitly, we have

$$\begin{aligned} \rho^{m_{\mathcal{A}} m_{\mathcal{B}}}_{n_{\mathcal{A}} n_{\mathcal{B}}}(t) &= \xi \sum_{\{k_i=1\}}^{\infty} \left\langle n_{\mathcal{A}}, k_1, k_2, \ldots \middle| \Gamma^{A}_{100\cdots}(t)\right\rangle \left\langle \Gamma^{A}_{100\cdots}(t) \middle| m_{\mathcal{A}}, k_1, k_2, \ldots\right\rangle \\ &\quad \times \sum_{\{q_i=1\}}^{\infty} \left\langle n_{\mathcal{B}}, q_1, q_2, \ldots \middle| \Gamma^{B}_{000\cdots}\right\rangle \left\langle \Gamma^{B}_{000\cdots} \middle| m_{\mathcal{B}}, q_1, q_2, \ldots\right\rangle \end{aligned}$$

$$
\begin{aligned}
&+(1-\xi)\sum_{\{k_i=1\}}^{\infty}\left\langle n_{\mathcal{A}},k_1,k_2,\ldots|\Gamma^A_{000\ldots}\right\rangle\left\langle\Gamma^A_{000\ldots}|m_{\mathcal{A}},k_1,k_2,\ldots\right\rangle\\
&\quad\times\sum_{\{q_i=1\}}^{\infty}\left\langle n_{\mathcal{B}},q_1,q_2,\ldots|\Gamma^B_{100\ldots}(t)\right\rangle\left\langle\Gamma^B_{100\ldots}(t)|m_{\mathcal{B}},q_1,q_2,\ldots\right\rangle\\
&+\sqrt{\xi(1-\xi)}e^{i\phi}\sum_{\{k_i=1\}}^{\infty}\left\langle n_{\mathcal{A}},k_1,k_2,\ldots|\Gamma^A_{000\ldots}\right\rangle\left\langle\Gamma^A_{100\ldots}(t)|m_{\mathcal{A}},k_1,k_2,\ldots\right\rangle\\
&\quad\times\sum_{\{q_i=1\}}^{\infty}\left\langle n_{\mathcal{B}},q_1,q_2,\ldots|\Gamma^B_{100\ldots}(t)\right\rangle\left\langle\Gamma^B_{000\ldots}|m_{\mathcal{B}},q_1,q_2,\ldots\right\rangle\\
&+\sqrt{\xi(1-\xi)}e^{-i\phi}\sum_{\{k_i=1\}}^{\infty}\left\langle n_{\mathcal{A}},k_1,k_2,\ldots|\Gamma^A_{100\ldots}(t)\right\rangle\left\langle\Gamma^A_{000\ldots}|m_{\mathcal{A}},k_1,k_2,\ldots\right\rangle\\
&\quad\times\sum_{\{q_i=1\}}^{\infty}\left\langle n_{\mathcal{B}},q_1,q_2,\ldots|\Gamma^B_{000\ldots}\right\rangle\left\langle\Gamma^B_{100\ldots}(t)|m_{\mathcal{B}},q_1,q_2,\ldots\right\rangle. \qquad (30)
\end{aligned}
$$

In the above expression we have typically

$$
\left\langle n_{\mathcal{A}},k_1,k_2,\ldots|\Gamma^A_{000\ldots}\right\rangle=\delta_{k_10}\delta_{k_20}\cdots \qquad (31)
$$

and

$$
\begin{aligned}
\left\langle n_{\mathcal{A}},k_1,k_2,\ldots|\Gamma^A_{100\ldots}(t)\right\rangle &= \sum_{\nu}f_{A\nu}(t)\left\langle n_{\mathcal{A}},k_1,k_2,\ldots|\Gamma^{\nu}_{100\ldots}(0)\right\rangle\\
&= f_{AA}(t)\left\langle n_{\mathcal{A}},k_1,k_2,\ldots|\Gamma^A_{100\ldots}(0)\right\rangle+\sum_{i=1}^{\infty}f_{Ai}(t)\left\langle n_{\mathcal{A}},k_1,k_2,\ldots|\Gamma^i_{100\ldots}(0)\right\rangle\\
&= f_{AA}(t)\delta_{n_{\mathcal{A}}1}\delta_{k_10}\delta_{k_20}\cdots+\sum_{i=1}^{\infty}f_{Ai}(t)\delta_{n_{\mathcal{A}}0}\delta_{k_10}\cdots\delta_{k_i1}\cdots \qquad (32)
\end{aligned}
$$

so that the sums in the elements of the reduced density matrix are of one of the types below:

$$
\begin{aligned}
\sum_{k_1,k_2,\ldots}\left\langle n_{\mathcal{A}},k_1,k_2,\ldots|\Gamma^A_{000\ldots}\right\rangle\left\langle\Gamma^A_{000\ldots}|m_{\mathcal{A}},k_1,k_2,\ldots\right\rangle &= \delta_{n_{\mathcal{A}}0}\delta_{m_{\mathcal{A}}0}\sum_{k_1}\delta_{k_10}\delta_{k_10}\sum_{k_2}\delta_{k_20}\delta_{k_20}\cdots\\
&= \delta_{n_{\mathcal{A}}0}\delta_{m_{\mathcal{A}}0}, \qquad (33)
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{k_1,k_2,\ldots}\left\langle n_{\mathcal{A}},k_1,k_2,\ldots|\Gamma^A_{000\ldots}\right\rangle\left\langle\Gamma^A_{100\ldots}(t)|m_{\mathcal{A}},k_1,k_2,\ldots\right\rangle\\
&= \delta_{n_{\mathcal{A}}0}\left[f^*_{AA}(t)\delta_{m_{\mathcal{A}}1}\sum_{k_1}\delta_{k_10}\delta_{k_10}\sum_{k_2}\delta_{k_20}\delta_{k_20}\cdots+\sum_i f^*_{Ai}(t)\sum_{k_1}\delta_{k_10}\delta_{k_10}\cdots\sum_{k_i}\delta_{k_i0}\delta_{k_i1}\cdots\right]\\
&= f^*_{AA}(t)\delta_{n_{\mathcal{A}}0}\delta_{m_{\mathcal{A}}1}, \qquad (34)
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{k_1,k_2,\ldots}\left\langle n_{\mathcal{A}},k_1,k_2,\ldots|\Gamma^A_{100\ldots}(t)\right\rangle\left\langle\Gamma^A_{100\ldots}(t)|m_{\mathcal{A}},k_1,k_2,\ldots\right\rangle\\
&= \sum_{k_1,k_2,\ldots}\left[f_{AA}(t)\delta_{n_{\mathcal{A}}1}\delta_{k_10}\delta_{k_20}\cdots+\sum_i f_{Ai}(t)\delta_{n_{\mathcal{A}}0}\delta_{k_10}\cdots\delta_{k_i1}\cdots\right]\\
&\qquad\times\left[f^*_{AA}(t)\delta_{m_{\mathcal{A}}1}\delta_{k_10}\delta_{k_20}\cdots+\sum_j f^*_{Aj}(t)\delta_{m_{\mathcal{A}}0}\delta_{k_10}\cdots\delta_{k_i1}\cdots\right]\\
&= |f_{AA}|^2\delta_{n_{\mathcal{A}}1}\delta_{m_{\mathcal{A}}1}+\sum_i|f_{Ai}|^2\delta_{n_{\mathcal{A}}0}\delta_{m_{\mathcal{A}}0}. \qquad (35)
\end{aligned}
$$

Collecting all these expressions and their analogues for the atom B into the elements of the reduced density matrix we finally obtain

$$\begin{aligned}\rho^{m_{\mathcal{A}}m_{\mathcal{B}}}_{n_{\mathcal{A}}n_{\mathcal{B}}}(t) &= \xi\left[|f_{AA}(t)|^2\,\delta_{n_{\mathcal{A}}1}\delta_{m_{\mathcal{A}}1}+\sum_i|f_{Ai}(t)|^2\,\delta_{n_{\mathcal{A}}0}\delta_{m_{\mathcal{A}}0}\right]\delta_{n_{\mathcal{B}}0}\delta_{m_{\mathcal{B}}0}\\ &+(1-\xi)\delta_{n_{\mathcal{A}}0}\delta_{m_{\mathcal{A}}0}\left[|f_{BB}(t)|^2\,\delta_{n_{\mathcal{B}}1}\delta_{m_{\mathcal{B}}1}+\sum_i|f_{Bi}(t)|^2\,\delta_{n_{\mathcal{B}}0}\delta_{m_{\mathcal{B}}0}\right]\\ &+\sqrt{\xi(1-\xi)}e^{i\phi}f^*_{AA}(t)f_{BB}(t)\delta_{n_{\mathcal{A}}0}\delta_{m_{\mathcal{A}}1}\delta_{n_{\mathcal{B}}1}\delta_{m_{\mathcal{B}}0}\\ &+\sqrt{\xi(1-\xi)}e^{-i\phi}f_{AA}(t)f^*_{BB}(t)\delta_{n_{\mathcal{A}}1}\delta_{m_{\mathcal{A}}0}\delta_{n_{\mathcal{B}}0}\delta_{m_{\mathcal{B}}1}.\end{aligned}\tag{36}$$

That is, the nonvanishing elements are given by

$$\begin{aligned}\rho^{0_{\mathcal{A}}0_{\mathcal{B}}}_{0_{\mathcal{A}}0_{\mathcal{B}}}(t) &= 1-\xi\,|f_{AA}(t)|^2-(1-\xi)\,|f_{BB}(t)|^2,\\ \rho^{0_{\mathcal{A}}1_{\mathcal{B}}}_{0_{\mathcal{A}}1_{\mathcal{B}}}(t) &= (1-\xi)\,|f_{BB}(t)|^2,\\ \rho^{1_{\mathcal{A}}0_{\mathcal{B}}}_{1_{\mathcal{A}}0_{\mathcal{B}}}(t) &= \xi\,|f_{AA}(t)|^2,\\ \rho^{1_{\mathcal{A}}0_{\mathcal{B}}}_{0_{\mathcal{A}}1_{\mathcal{B}}}(t) &= \sqrt{\xi(1-\xi)}e^{i\phi}f^*_{AA}(t)f_{BB}(t),\\ \rho^{0_{\mathcal{A}}1_{\mathcal{B}}}_{1_{\mathcal{A}}0_{\mathcal{B}}}(t) &= \sqrt{\xi(1-\xi)}e^{-i\phi}f_{AA}(t)f^*_{BB}(t),\end{aligned}\tag{37}$$

where equation (22) was used. We check immediately that the trace of this reduced density matrix is one,

$$\rho^{0_{\mathcal{A}}0_{\mathcal{B}}}_{0_{\mathcal{A}}0_{\mathcal{B}}}+\rho^{0_{\mathcal{A}}1_{\mathcal{B}}}_{0_{\mathcal{A}}1_{\mathcal{B}}}+\rho^{1_{\mathcal{A}}0_{\mathcal{B}}}_{1_{\mathcal{A}}0_{\mathcal{B}}}(t)+\rho^{1_{\mathcal{A}}1_{\mathcal{B}}}_{1_{\mathcal{A}}1_{\mathcal{B}}}(t)=1.\tag{38}$$

This property ensures that ρ represents physical states of the system. Also, we see that $\mathrm{Tr}\left[\rho^2\right]\neq 1$ and therefore, the superposed states are not pure. The degree of impurity of a quantum state can be quantified by the departure from the idempotency property. In the present case:

$$\begin{aligned}D(t,\xi) &= 1-\mathrm{Tr}\left[\rho^2\right]\\ &= 2\left(\xi\,|f_{AA}(t)|^2+(1-\xi)\,|f_{BB}(t)|^2\right)-2\left(\xi\,|f_{AA}(t)|^2+(1-\xi)\,|f_{BB}(t)|^2\right)^2.\end{aligned}\tag{39}$$

In the remainder of this section we consider the two atoms as identical and, accordingly, we adopt the subscript 0 for both of them, $\lambda=A=B\equiv 0$; we also define

$$g_A=g_B\equiv g\,;\;\;\eta_A=\eta_B\equiv\eta\,;\;\;\bar{\omega}_A=\bar{\omega}_B\equiv\bar{\omega}\,;\;\;f_{AA}(t)=f_{BB}(t)\equiv f_{00}(t).\tag{40}$$

In this case, the matrix elements in equations (37) simplify and, from equation (39), we see that the degree of impurity becomes independent of the superposition parameter ξ:

$$D(t,\xi)=2\,|f_{00}(t)|^2\,(1-|f_{00}(t)|^2).\tag{41}$$

In order to pursue the study of the time evolution of the superposition of the two-atom states, we have to determine the behavior of $f_{00}(t)$. We shall analyze it in the situations of a very large cavity (free space) and of a small one.

A. The limit of an arbitrarily large cavity

We start from the matrix element $t^{r\lambda}_{\lambda}$ in equation (14) and consider an arbitrarily large radius R for the cavity. The two atoms behave independently from each other, so let us focus on just one of them, either the atom A or the atom B. Remembering that $\eta=\sqrt{4gc/R}$, we have

$$\lim_{R\to\infty}t^r_0=\lim_{R\to\infty}\frac{\sqrt{4g/\pi}\Omega\sqrt{\pi c/R}}{\sqrt{\left(\Omega^2-\bar{\omega}^2\right)^2+4g^2\Omega^2}}.\tag{42}$$

In this limit, $\Delta\omega=\pi c/R\to d\omega=d\Omega$ and the sum in the definition of $f_{00}(t)$, equation (21), becomes an integral, so that

$$f_{00}(t)=\frac{4g}{\pi}\int_0^\infty d\Omega\frac{\Omega^2e^{-i\Omega t}}{\left(\Omega^2-\bar{\omega}^2\right)^2+4g^2\Omega^2}.\tag{43}$$

We then proceed as in [24]. We define a parameter $\kappa = \sqrt{\bar{\omega}^2 - g^2}$ and consider whether $\kappa^2 \geq 0$ or $\kappa^2 < 0$, for which $\kappa^2 \gg 0$ and $\kappa^2 \ll 0$ correspond respectively to weak ($g \ll \bar{\omega}_A$) and strong ($g \gg \bar{\omega}_A$) coupling of the atoms with the environmment. For definiteness we consider in the following the weak-coupling regime. We get in this case [24]

$$f_{00}(t) = e^{-gt}\left[\cos\kappa t - \frac{g}{\kappa}\sin\kappa t\right] + iG\left(t;\bar{\omega},g\right), \tag{44}$$

where the function $G(t;\bar{\omega},g)$ is given by

$$G(t;\bar{\omega},g) = -\frac{4g}{\pi}\int_0^\infty dx \frac{x^2 \sin xt}{\left(x^2-\bar{\omega}^2\right)^2 + 4g^2x^2}. \tag{45}$$

For large times, the quantity $\left|f_{00}(t)\right|^2$ is given by [24]

$$\left|f_{00}(t)\right|^2 \approx e^{-2gt}\left[\cos\bar{\omega}t - \frac{g}{\bar{\omega}}\sin\bar{\omega}t\right]^2 + \frac{64g^2}{\bar{\omega}^8 t^6}. \tag{46}$$

As $t \to \infty$, we see that the expression for $\left|f_{00}(t)\right|^2$ go to zero.

B. Small cavity

For a finite (small) cavity, the spectrum of eigenfrequencies is discrete, $\Delta\omega$ is large, and so the approximation made in the case of large cavity does not apply; no analytical result can be obtained for $f_{00}(t)$ in this case. For a sufficiently small cavity, the frequencies Ω_r can be determined by following the steps described in [24]. Let us label the eigenfrequencies as Ω_0, $\{\Omega_k\}$, $k = 1, 2, \ldots$ Then, defining the dimensionless parameter

$$\delta = \frac{g}{\Delta\omega} = \frac{gR}{\pi c}, \tag{47}$$

we rewrite equation (13) in the form

$$\cot\left(\frac{R\Omega_r}{c}\right) = \frac{\Omega_r}{2g} + \frac{c}{R\Omega_r}\left(1 - \frac{R\bar{\omega}^2}{2gc}\right). \tag{48}$$

Taking $\delta \ll 1$, which corresponds to $R \ll \pi c/g$ (a small cavity), it is shown in [24] that, for $k = 1, 2, \ldots$, the solutions are

$$\Omega_k \approx \frac{g}{\delta}\left(k + \frac{2\delta}{\pi k}\right). \tag{49}$$

If we further impose that $\delta < 2g^2/\pi\bar{\omega}^2$, a condition compatible with $\delta \ll 1$, then Ω_0 is found to be very close to $\bar{\omega}$, that is,

$$\Omega_0 \approx \bar{\omega}\left(1 - \frac{\pi\delta}{3}\right). \tag{50}$$

To determine $f_{00}(t)$, we have to calculate the square of the matrix elements $\left(t_0^0\right)^2$ and $\left(t_k^0\right)^2$. They are given, to first order in δ, by

$$\left(t_0^0\right)^2 \approx \left(1 + \frac{2\pi\delta}{3}\right)^{-1}; \qquad \left(t_k^0\right)^2 \approx \frac{4}{k^2}\frac{\delta}{\pi}(t_0^0)^2. \tag{51}$$

We thus obtain, for sufficiently small cavities ($\delta \ll 1$),

$$\begin{aligned}\left|f_{00}(t)\right|^2 \;\approx\; & \left(1 + \frac{2}{3}\pi\delta\right)^{-2}\left\{1 + \frac{8\delta}{\pi}\sum_{k=1}^{\infty}\frac{1}{k^2}\cos\left[\bar{\omega}\left(1 - \frac{\pi\delta}{3}\right) - \frac{g}{\delta}\left(k + \frac{2\delta}{\pi k}\right)\right]t\right. \\ & \left. + \frac{16\delta^2}{\pi^2}\sum_{k,l=1}^{\infty}\frac{1}{k^2l^2}\cos\left[\left(\frac{g}{\delta} - \frac{2g}{\pi kl}\right)(k-l)\right]t\right\}. \end{aligned} \tag{52}$$

To order δ^2, a lower bound for $|f_{00}(t)|^2$ is obtained by taking the value -1 for both cosines in the above formula, using the tabulated value of the Riemann zeta function $\zeta(2) = \pi^2/6$,

$$|f_{00}(t)|^2 \gtrsim \left(1 + \frac{2}{3}\pi\delta\right)^{-2} \left\{1 - \frac{4\pi\delta}{3} - \frac{4\pi^2\delta^2}{9}\right\}. \tag{53}$$

We see that the quantity $|f_{00}(t)|^2$, which dictates the behavior of the density matrix elements and of the measure of purity in equation (41), has very different behaviors for free space or for a small cavity. This implies that in the situation of a small cavity, in contrast to the free space case, all matrix elements in equations (37) are different from *zero* for all times.

In Figure (1) the degree of impurity from equation (41) is plotted as a function of time in the cases of an arbitrarily large cavity ($R \to \infty$) and of a small cavity. We take $\delta = 0.1$, with $\bar{\omega} = 1.0$ and $g = 0.5$ fixed (in arbitrary units).

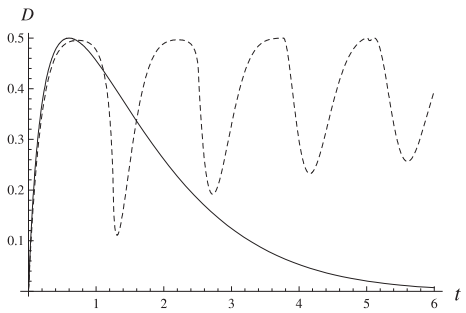

FIG. 1: Behavior of the degree of impurity D as function of time, equation (41), for a small cavity (dashed line) and a very large cavity (solid line); we take the parameters $g = 0.5$, $\delta = 0.1$ and $\bar{\omega} = 1.0$ (in arbitrary units).

We see from the figure that for a very large cavity (free space) the two-atom system dissipates; with the passing of time, both atoms go to their ground states. For a small cavity the system never completely decay.

IV. TIME EVOLUTION OF THE ENTANGLEMENT

In order to study how the entanglement of the two-atom states evolves in time, we shall, as before, consider the system as a bipartite system, in which each atom carries its own dressing field. In this way, we shall determine the time behavior of the von Neumann entropy associated with the reduced density matrix with respect to one of the subsystems, which is given by taking the trace over the states of the complementary subsystem.

Let us initially treat the system at $t = 0$. Then we have, for the subsystem $\mathcal{A}$, the reduced density matrix

$$\begin{aligned}
\rho_{\mathcal{A}}(0) &= \mathrm{Tr}_{\mathcal{B}}\left(|\Psi\rangle\langle\Psi|\right) \\
&= \sum_{n_{\mathcal{B}},q_j=0}^{\infty} \langle n_{\mathcal{B}}, q_1, q_2, \ldots |\Psi\rangle \langle\Psi| n_{\mathcal{B}}, q_1, q_2, \ldots\rangle \\
&= \xi\,|1_A, 0, 0, \cdots\rangle\langle 1_A, 0, 0, \cdots| + (1-\xi)\,|0_A, 0, 0, \cdots\rangle\langle 0_A, 0, 0, \cdots|
\end{aligned} \tag{54}$$

and, similarly, for the subsystem $\mathcal{B}$,

$$\begin{aligned}
\rho_{\mathcal{B}}(0) &= \mathrm{Tr}_{\mathcal{A}}\left(|\Psi\rangle\langle\Psi|\right) \\
&= \xi\,|0_B, 0, 0, \cdots\rangle\langle 0_B, 0, 0, \cdots| + (1-\xi)\,|1_B, 0, 0, \cdots\rangle\langle 1_B, 0, 0, \cdots|\,.
\end{aligned} \tag{55}$$

The degree of entanglement of the two-atom system is measured by the von Neumann entropy of any of the reduced density matrices; for instance,

$$E(\xi) = -\mathrm{Tr}\left[\rho_{\mathcal{A}} \ln \rho_{\mathcal{A}}\right] = -\sum_{\alpha} \alpha \ln \alpha, \tag{56}$$

where the sum is taken over the eigenvalues α of $\rho_{\mathcal{A}}$. Since $\rho_{\mathcal{A}}$ is diagonal in the Fock basis of the dressed states of the atom $\mathcal{A}$, its eigenvalues can be read directly from (54):

$$\alpha_1 = 1 - \xi, \qquad \alpha_2 = \xi, \qquad \alpha_2 = \alpha_3 = \cdots = 0. \tag{57}$$

Therefore,

$$E(\xi) = -\left[(1-\xi)\ln(1-\xi) + \xi \ln \xi\right]. \tag{58}$$

The time evolution of the states $\left|\Gamma^{(AB)}_{1(A)0(B)00\cdots}\right\rangle$ and $\left|\Gamma^{(AB)}_{0(A)1(B)00\cdots}\right\rangle$ are governed by the time evolution of the states $\left|\Gamma^{A}_{100\cdots}\right\rangle$ and $\left|\Gamma^{B}_{100\cdots}\right\rangle$, respectively, given by equation (20),

$$\left|\Gamma^{\lambda}_{100\cdots}(t)\right\rangle = \sum_{\nu} f_{\lambda\nu}(t) \left|\Gamma^{\nu(\lambda)}_{100\cdots}\right\rangle, \tag{59}$$

where, in accord with the notation of the preceding section, the label λ now refers to each one of the dressed atoms A and B and

$$f_{\lambda\nu}(t) = \sum_{s} t^{s}_{\lambda} t^{s}_{\nu} e^{-i\Omega_s t}. \tag{60}$$

In equation (59), $\left|\Gamma^{\nu(\lambda)}_{100\cdots}\right\rangle$ is the state in which the dressed mode $\nu(\lambda)$ of the atom λ is at the first level and all the other dressed modes are in the ground state.

The reduced density matrix corresponding to the subsystem $\mathcal{A}$ at time t is

$$\rho_{\mathcal{A}}(t) = \mathrm{Tr}_{\mathcal{B}}\varrho(t) = \mathrm{Tr}_{\mathcal{B}}\left[|\Psi(t)\rangle\langle\Psi(t)|\right].$$

Using equation (59), one writes $\rho_{\mathcal{A}}(t)$ in terms of the quantities $f_{\lambda\nu}(t)$ from equation (60):

$$\begin{aligned}
\rho_{\mathcal{A}}(t) &= \sum_{n_{\mathcal{B}},q_j=0}^{\infty} \langle n_{\mathcal{B}}, q_1, q_2, \ldots |\Psi(t)\rangle \langle \Psi(t)| n_{\mathcal{B}}, q_1, q_2, \ldots\rangle \\
&= \sum_{n_{\mathcal{B}},q_j}\sum_{\mu,\nu}\left[\sqrt{\xi} f_{A\mu}(t)\left|\Gamma^{\mu(A)}_{100\cdots}\right\rangle\langle n_{\mathcal{B}}, q_1, q_2, \ldots \left|\Gamma^{B}_{000\cdots}\right\rangle + \sqrt{1-\xi}\, e^{i\phi} f_{B\mu}(t)\left|\Gamma^{A}_{000\cdots}\right\rangle\left\langle n_{\mathcal{B}}, q_1, q_2, \ldots \right|\Gamma^{\mu(B)}_{100\cdots}\right\rangle\right] \\
&\quad \times\left[\sqrt{\xi} f^{*}_{A\nu}(t)\left\langle\Gamma^{\nu(A)}_{100\cdots}\right|\left\langle\Gamma^{B}_{000\cdots}\right| n_{\mathcal{B}}, q_1, q_2, \ldots\rangle + \sqrt{1-\xi}\, e^{-i\phi} f^{*}_{B\nu}(t)\langle\Gamma^{A}_{000\cdots}|\left\langle\Gamma^{\nu(B)}_{100\cdots}\right| n_{\mathcal{B}}, q_1, q_2, \ldots\right\rangle\right] \\
&= \sum_{\mu,\nu}\xi f_{A\mu}(t) f^{*}_{A\nu}(t)\left|\Gamma^{\mu(A)}_{100\cdots}\right\rangle\left\langle\Gamma^{\nu(A)}_{100\cdots}\right| + (1-\xi)\left|\Gamma^{A}_{000\cdots}\right\rangle\langle\Gamma^{A}_{000\cdots}|,
\end{aligned} \tag{61}$$

where we have used

$$\sum_{\mu(\lambda)} f_{B\mu}(t)\left\langle n_{\mathcal{B}}, q_1, q_2, \ldots\right|\Gamma^{\mu(B)}_{100\cdots}\rangle = f_{BB}(t)\delta_{n_{\mathcal{B}}1}\prod_i \delta_{i0} + \delta_{n_{\mathcal{B}}0}\sum_i f_{Bi}(t)\delta_{i1}\prod_{j\neq i}\delta_{j0} \tag{62}$$

and equation (22).

The time-dependent von Neumann entropy is now given by

$$E(t,\xi) = -\mathrm{Tr}\left[\rho_{\mathcal{A}}(t)\ln\rho_{\mathcal{A}}(t)\right] = -\sum_{\alpha}\alpha\ln\alpha, \tag{63}$$

where here α are the time-dependent eigenvalues of the reduced density matrix. These should be solutions of the so-called characteristic equation, which in the case of (61), reads

$$\det\begin{pmatrix}
1-\xi-\alpha & & & & \\
& \xi|f_{AA}|^2-\alpha & \xi f_{A1}f^{*}_{AA} & \xi f_{A2}f^{*}_{AA} & \cdots \\
& \xi f_{AA}f^{*}_{A1} & \xi|f_{A1}|^2-\alpha & \xi f_{A2}f^{*}_{A1} & \cdots \\
& \xi f_{AA}f^{*}_{A2} & \xi f_{A1}f^{*}_{A2} & \xi|f_{A2}|^2-\alpha & \cdots \\
& \vdots & \vdots & \vdots & \ddots
\end{pmatrix} = 0. \tag{64}$$

We thus find that the nonzero eigenvalues of $\rho_{\mathcal{A}}$ are

$$\alpha_1 = 1-\xi, \qquad \alpha_2 = \xi\sum_{\mu(\lambda)}|f_{A\mu}(t)|^2 = \xi. \tag{65}$$

This then implies that the von Neumann entropy takes the expression

$$E(t,\xi) = -\left[(1-\xi)\ln(1-\xi) + \xi \ln(\xi)\right], \tag{66}$$

that is, all the time dependence of the von Neumann entropy for this two-atom system, coming from the $f_{\lambda\nu}(t)$, is completely cancelled in the computation of the entropy, in all situations, thereby reproducing exactly the same expression as in the $t = 0$ case, with the maximum entanglement occuring at $\xi = 1/2$ (see Figure 2). In other words, although the superposition of states evolves in time, in different ways in the limits of a very large cavity and of a small one, the entangled nature of these two-atom states remains unchanged for all times, independently of the size of the cavity.

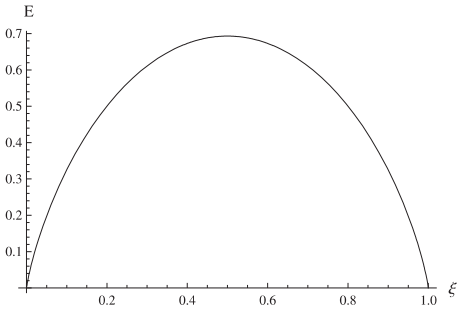

FIG. 2: Behaviour of the von Neumann entropy at all times, equation (66), as a function of the parameter ξ.

V. CONCLUDING REMARKS

In this paper we have considered a system composed of two atoms in a spherical cavity, each of them in independent interaction with an environment field. The model employed is of a bipartite system, in which each subsystem consists of one of the atoms dressed by the field. We make the assumption that initially we have a state in which one of the dressed atoms is in its first excited level and the other atom and the field modes are all in the ground state, is superposed with a state in which the atoms have their roles reversed.

The time evolution of the superposed states leads to a time-dependent (reduced) density matrix. Expressions for its elements are provided in both the cases of an infinitely large cavity (that is, free space) and of a small one, when the two atoms are considered as identical. Very different behaviors are obtained for this time evolution. In the large-cavity case, the system shows dissipation, and, with the passing of time, both atoms go to their ground states. For a small cavity, an oscillating behavior is present, so that the atoms never fully decay.

Nevertheless, in spite of these rather contrasting behaviors and of the nontrivial time dependence of the density matrix, we obtain a time-independent von Neumann entropy, which means that the initial entanglement of the two atoms remains unchanged as the system evolves.

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